

变系数热传导方程反初值问题的拟边值方法

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摘要:针对一维区域上带有时间依赖系数的非齐次热传导方程的反初值问题,采用拟边值方法求解此问题。首先根据分离变量法得到问题的解,并根据问题解的表达式构造了正则化解;其次在原问题的解满足某些先验条件下,给出正则化参数选取的先验和后验方法,并在理论上严格证明了在此参数选取准则下,一维热传导方程反初值问题正则化解的收敛性;最后通过数值模拟表明,拟边值方法对于求解此反初值问题是有效和稳定的。

关键词:不适定问题;正则化参数;拟边值方法;热传导方程;误差估计

DOI:10.15938/j.jhust.2023.05.017

中图分类号: O241.8 **文献标志码:** A **文章编号:** 1007-2683(2023)05-0136-07

Quasi-boundary Value Method for Backward Heat Conduction Equation with Variable Coefficients

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Abstract: Aiming at the problem of inhomogeneous backward heat conduction equation with time-dependent coefficients in a one-dimensional region, the quasi-boundary value method is used to solve this problem. Firstly, the solution of the problem is obtained by separating variables, and according to the expression of the solution of the problem, the regular solution is constructed; secondly, when the solution of the original problem satisfies some prior conditions, the priori and posteriori methods for the regularization parameter are given respectively, and the convergence of the regularization solution of the problem of one-dimensional backward heat conduction equation under this parameter selection criterion is strictly proved; finally, numerical simulation shows that quasi-boundary value method is effective and stable.

Keywords: ill-posed problem; regularization parameter; quasi-boundary value method; heat conduction equation; error estimation

0 引言

热传导方程的反初值问题也被称为逆时问题,是热传导方程反问题的一种,在很多实际问题中有着广泛应用,例如热流、遥感技术、航天防护服表面温度控制等。此类问题是一个不适定问题^[1],很难用传统的方法来解决,为了得到稳定的近似解,国内外学者对这类问题进行了研究,提出了很多方法,例

如 Tikhonov 正则化方法^[2-3]、滤波正则化方法^[4]、拟逆方法^[5-8]、拟边值方法^[9-18]等。但目前大部分研究还是集中在齐次方程且测量数据只有一个,或者常系数非齐次方程的反初值问题,对变系数非齐次方程的研究较少,并且大部分研究只给出了正则化参数的先验选取策略,针对后验选取规则的研究也很少。

本文考虑如下一个一维带有时间依赖系数的非齐次热传导方程的反初值问题,

收稿日期: 2022-05-22

基金项目: 黑龙江省自然科学基金(LH2020A015)。

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$$\left. \begin{aligned} u_t(x,t) - a(t)u_{xx}(x,t) &= f(x,t), 0 \leq x \leq L, 0 \leq t \leq T \\ u(0,t) = u(L,t) &= 0, 0 \leq t \leq T \\ u(x,T) &= g(x), 0 \leq x \leq L \end{aligned} \right\} \quad (1)$$

其中 $f(x,t)$ 是关于 t 的连续可微函数且 $f(x,t) \in L^\infty(0,T;L^2[0,L])$, $g(x) \in L^2[0,L]$, $a(t) \in C^\infty[0,T]$, 且存在正常数 m 和 d ,使得

$$0 < m \leq a(t) \leq d$$

反初值问题就是利用 $g(x)$ 和 $f(x,t)$ 带有噪声的测量数据 $g^\delta(x)$, $f^\delta(x,t)$, 求解 $u(x,0) =: \varphi(x)$ 。假设测量数据满足条件:

$$\begin{aligned} \|g^\delta(x) - g(x)\|_{L^2[0,L]} &\leq \delta \\ \|f^\delta(x,t) - f(x,t)\|_{L^\infty(0,T;L^2[0,L])} &\leq \delta \end{aligned}$$

本文使用拟边值方法求解这个问题, 拟边值方法也称为非局部边值方法, 是一种用新的近似条件代替终值条件或边界条件的正则化方法, 最早由Showalter^[9]提出。拟边值方法已经被应用于求解各种类型方程的反问题中, 例如, 分数阶扩散方程^[19-20]、非线性抛物方程^[21]、椭圆型方程^[16]等。

Triet Minh Le 等在文[14]中提出了一种改进的正则化方法来求解问题(1), 并给出了一种特殊情况下的正则化参数的先验选取规则。马宗立等^[22]把这一方法推广到二维圆域上给出了误差估计。除此之外, 还未见到其他文献研究此问题。本文所用的拟边值方法在求解 $u(x,0)$ 时, 与文[14]的方法是等价的, 但是这两种方法计算正则化解的方式是不同的; 文[14]中的正则化解的计算需要用到椭圆算子的特征值和特征函数, 所以很难将其推广到高维一般区域中, 而本文所使用的拟边值方法可以实现。本文不仅给出了正则化参数的先验选取规则, 也研究了当解满足某种先验条件时的后验选取策略, 根据相关引理和定理推导了正则化解和精确解的误差估计, 最后用数值算例验证本文所采用的拟边值方法求解变系数热传导方程的反初值问题具有可行性。

下面用 $(\cdot, \cdot)$ 和 $\|\cdot\|$ 分别表示 $L^2[0,L]$ 上的内积和范数。根据分离变量法, 问题(1)的解形式上可以表示为

$$u(x,t) = \sum_{n=1}^{\infty} (e^{-\lambda_n A(t)} \varphi_n + \int_0^t e^{-\lambda_n(A(t)-A(\tau))} f_n(\tau) d\tau) \omega_n(x)$$

$$\begin{aligned} \text{其中 } A(t) &= \int_0^t a(s) ds, \omega_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}, \lambda_n = \\ &= \frac{n^2 \pi^2}{L^2} \end{aligned}$$

再令上式中 $t = T$, 得到

$$g(x) = \sum_{n=1}^{\infty} (e^{-\lambda_n A(T)} \varphi_n + \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau) \omega_n(x)$$

故

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{g_n - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau}{e^{-\lambda_n A(T)}} \omega_n(x) \quad (2)$$

当 $n \rightarrow \infty$ 时, $\lambda_n \rightarrow \infty$, $(e^{-\lambda_n A(T)})^{-1} \rightarrow \infty$, 故问题(1)是不适定的。

定义算子 $K^{[23]}: L^2[0,L] \rightarrow L^2[0,L]$,

$$(K\varphi)(x) = \int_0^L k(x,\xi) \varphi(\xi) d\xi$$

其中 $k(x,\xi) = \sum_{n=1}^{\infty} e^{-\lambda_n A(T)} \omega_n(x) \omega_n(\xi)$, 不难看出 $K: L^2[0,L] \rightarrow L^2[0,L]$ 是一个线性自伴紧算子。

1 拟边值方法

采用拟边值方法求解问题(1), 修改(1)中的终值条件 $u(x,T) = g(x)$, 得到如下问题:

$$\left. \begin{aligned} u_t^\mu(x,t) - a(t)u_{xx}^\mu(x,t) &= f^\delta(x,t), 0 \leq x \leq L, 0 \leq t \leq T \\ u^\mu(0,t) = u^\mu(L,t) &= 0, 0 \leq t \leq T \\ u^\mu(x,T) + \mu u^\mu(x,0) &= g^\delta(x), 0 \leq x \leq L \end{aligned} \right\} \quad (3)$$

其中 μ 为正则化参数。利用分离变量法得到,

$$\varphi^{\mu\delta}(x) = u^{\mu\delta}(x,0) = \sum_{n=1}^{\infty} \frac{g_n^\delta - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n^\delta(\tau) d\tau}{\mu + e^{-\lambda_n A(T)}} \omega_n(x) \quad (4)$$

上式即为拟边值方法构造的正则化解。定义

$$\varphi^\mu(x) = \sum_{n=1}^{\infty} \frac{g_n - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau}{\mu + e^{-\lambda_n A(T)}} \omega_n(x) \quad (5)$$

1.1 正则化参数的先验选取规则

引理1^[22] 令 $\eta > 0, 0 \leq a \leq b$, 则对任意的 $k > 0$, 都有

$$\frac{e^{ka}}{1 + \eta e^{kb}} \leq \eta^{-\frac{a}{b}}$$

引理2 若 $p \geq 2$, 则对任意的 n , 有

$$\frac{\mu(e^{\lambda_n A(T)})^{\frac{2-p}{4}}}{1 + \mu e^{\lambda_n A(T)}} \leq C_2 \mu$$

其中 C_2 是与 n 无关的常数。

证明: 由 $p \geq 2$, 易得

$$\frac{\mu(e^{\lambda_n A(T)})^{\frac{2-p}{4}}}{1 + \mu e^{\lambda_n A(T)}} \leq \mu (e^{\lambda_n A(T)})^{\frac{2-p}{4}} = \frac{\mu}{(e^{\lambda_n A(T)})^{\frac{p-2}{4}}} \leq$$

$$\frac{\mu}{(e^{\lambda_1 A(T)})^{\frac{p-2}{4}}} \leq C_2 \mu$$

引理3 给定 $\sum_{n=1}^{\infty} g_n - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau$ 和

$f(x, t) \in L^\infty(0, T; L^2[0, L])$, 有

$$\left\| \sum_{n=1}^{\infty} \left(g_n - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau \right) \omega_n(x) \right\| \leq \sqrt{2} (\|g\|^2 + M \|f\|_{L^\infty(0, T; L^2[0, L])}^2) \quad \text{其中 } M = \sum_{n=1}^{\infty} \left(\int_0^T e^{-\lambda_n(A(T)-A(\tau))} d\tau \right)^2.$$

证明:与文[16]中引理2.4类似。

定理1 若 $0 < \mu < 1$, 问题(1)有精确解 $\varphi(x)$,

1) 如果存在常数 p 和 E_1 , 满足 $(\sum_{n=1}^{\infty} \lambda_n^p \varphi_n^2)^{\frac{1}{2}} \leq E_1 < +\infty$, 则有

$$\|\varphi^\mu(x) - \varphi(x)\| \leq C_1 E_1 \left(\frac{A(T)}{-\ln \mu} \right)^{\frac{p}{2}} (o(1) +$$

$1)(\mu \rightarrow 0^+)$

2) 如果存在常数 p 和 E_2 , 满足 $(\sum_{n=1}^{\infty} e^{p\lambda_n A(T)} \varphi_n^2)^{\frac{1}{2}} \leq E_2 < +\infty$, 则有

$$\|\varphi^\mu(x) - \varphi(x)\| \leq \begin{cases} \mu^{\frac{p}{2}} E_2, & 0 < p < 2 \\ C_2 \mu E_2, & p \geq 2 \end{cases}$$

其中 C_1, C_2 为正常数。

证明

$$\|\varphi^\mu(x) - \varphi(x)\| =$$

$$\left(\sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \left(g_n - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau \right) \right)^2 \right)^{\frac{1}{2}}$$

若 $\varphi(x)$ 满足条件1), 则,

$$\|\varphi^\mu(x) - \varphi(x)\|^2 = \sum_{n=1}^{\infty} \frac{\mu^2 \lambda_n^{-p}}{(\mu + e^{-\lambda_n A(T)})^2} \lambda_n^p \varphi_n^2$$

当 $\lambda_n A(T) \leq 1$ 时, 令

$$S_1 = \sum_{\lambda_n A(T) \leq 1} \frac{\mu^2 \lambda_n^p \varphi_n^2}{\lambda_n^p (\mu + e^{-\lambda_n A(T)})^2} \leq \sum_{\lambda_n A(T) \leq 1} \mu^2 \lambda_n^{-p} e^{2\lambda_n A(T)} (\lambda_n^p \varphi_n^2) \leq \sum_{\lambda_n A(T) \leq 1} \mu^2 \lambda_1^{-p} e^2 (\lambda_n^p \varphi_n^2)$$

当 $\lambda_n A(T) > 1$ 时, 令 $\mu Z_n = e^{-\lambda_n A(T)} < 1$, 则 $\lambda_n = -\frac{\ln(\mu Z_n)}{A(T)}$,

令

$$S_2 = \sum_{\lambda_n A(T) > 1} \frac{\mu^2 \lambda_n^p \varphi_n^2}{\lambda_n^p (\mu + e^{-\lambda_n A(T)})^2} = \sum_{\lambda_n A(T) > 1} \frac{\mu^2}{(\mu + \mu Z_n)^2} \left(-\frac{\ln(\mu Z_n)}{A(T)} \right)^{-p} \lambda_n^p \varphi_n^2 = \sum_{\lambda_n A(T) > 1} \left(-\frac{A(T)}{\ln \mu} \right)^p \frac{1}{(1 + Z_n)^2} \left(\frac{\ln \mu}{\ln(\mu Z_n)} \right)^p \lambda_n^p \varphi_n^2$$

再令 $\gamma_n = \frac{1}{(1 + Z_n)^2} \left(\frac{\ln \mu}{\ln(\mu Z_n)} \right)^p$, 下证 γ_n 一致有界。

若 $0 < Z_n \leq 1$, 则 $\ln(\mu Z_n) \leq \ln \mu < 0$,

$$\gamma_n = \frac{1}{(1 + Z_n)^2} \left(\frac{\ln \mu}{\ln(\mu Z_n)} \right)^p \leq 1$$

若 $Z_n > 1$, 则 $\ln Z_n > 0, \ln(\mu Z_n) = -\lambda_n A(T) < -1$, 接下来有

$$0 < \frac{\ln \mu}{\ln(\mu Z_n)} = 1 - \frac{\ln Z_n}{\ln(\mu Z_n)} < 1 + \ln Z_n$$

则

$$\gamma_n \leq \frac{(1 + \ln Z_n)^p}{(1 + Z_n)^2} \leq \frac{(1 + \ln Z_n)^p}{1 + Z_n} < \frac{(1 + \ln Z_n)^p}{Z_n} =: q(Z_n)$$

又因为 $q'(Z_n) = \frac{p(1 + \ln Z_n)^{p-1} - (1 + \ln Z_n)^p}{Z_n^2}$, 故

$\gamma_n \leq q_{\max}(Z_n^*) = p^p e^{1-p} =: Q_p$, 所以 γ_n 一致有界。当 $\mu \rightarrow 0^+$, 则

$$\|\varphi^\mu(x) - \varphi(x)\|^2 \leq (\mu^2 \lambda_1^{-p} e^2 + Q_p \left(-\frac{A(T)}{\ln \mu} \right)^p) \sum_{n=1}^{\infty} \lambda_n^p \varphi_n^2 \leq$$

$$C_1^2 E_1^2 \left(-\frac{A(T)}{\ln \mu} \right)^p (o(1) + 1)$$

其中 $C_1^2 = \max\{\lambda_1^{-p} e^2, Q_p\}$ 。

若 $\varphi(x)$ 满足条件(2), 有

$$\|\varphi^\mu(x) - \varphi(x)\| = \left(\sum_{n=1}^{\infty} \left(\frac{\mu e^{-\frac{\lambda_n p}{2} A(T)}}{\mu + e^{-\lambda_n A(T)}} e^{\frac{\lambda_n p}{2} A(T)} \varphi_n \right)^2 \right)^{\frac{1}{2}}$$

再由引理1和引理2, 得

$$\|\varphi^\mu(x) - \varphi(x)\| \leq \begin{cases} \mu^{\frac{p}{2}} E_2, & 0 < p < 2 \\ C_3 \mu E_2, & p \geq 2 \end{cases} \quad (6)$$

定理得证。

定理2 在定理1的条件下,

1) 若存在常数 p 和 E_1 , 满足 $(\sum_{n=1}^{\infty} \lambda_n^p \varphi_n^2)^{\frac{1}{2}} \leq$

$E_1 < +\infty$, 令 $\mu = \left(\frac{\delta}{E_1} \right) \left(\frac{1}{A(T)} \ln \frac{E_1}{\delta} \right)^{\frac{p}{2}}$, 有

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq (\sqrt{2(M+1)} + C_1)$$

$$E_1 \left(\frac{1}{A(T)} \ln \frac{E_1}{\delta} \right)^{-\frac{p}{2}} (o(1) + 1) (\delta \rightarrow 0^+)$$

2) 若存在常数 p 和 E_2 , 满足 $(\sum_{n=1}^{\infty} e^{\rho \lambda_n A(T)} \varphi_n^2)^{\frac{1}{2}} \leq E_2 < +\infty$, 有

(a) 当 $0 < p < 2$ 时, 令 $\mu = \left(\frac{\delta}{E_2} \right)^{\frac{2}{p+2}}$, 则

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq (\sqrt{2(M+1)} + 1) \delta^{\frac{p}{p+2}} E_2^{\frac{2}{p+2}}$$

(b) 当 $p \geq 2$ 时, 令 $\mu = \left(\frac{\delta}{E_2} \right)^{\frac{1}{2}}$, 则

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq (\sqrt{2(M+1)} + C_2) E_2^{\frac{1}{2}} \delta^{\frac{1}{2}}$$

证明

$$\begin{aligned} \|\varphi^{\mu, \delta}(x) - \varphi^{\mu}(x)\|^2 &= \left\| \sum_{n=1}^{\infty} \left(\frac{g_n^{\delta} - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n^{\delta}(\tau) d\tau}{\mu + e^{-\lambda_n A(T)}} \right) - \right. \\ &\quad \left. \frac{g_n - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau}{\mu + e^{-\lambda_n A(T)}} \omega_n(x) \right\|^2 \leq \\ &2 \sum_{n=1}^{\infty} \left(\frac{(g_n^{\delta} - g_n)^2 + \int_0^T e^{-\lambda_n(A(T)-A(\tau))} (f_n^{\delta}(\tau) - f_n(\tau))^2 d\tau}{(\mu + e^{-\lambda_n A(T)})^2} \right) \leq \\ &2 \sum_{n=1}^{\infty} \left(\frac{(g_n^{\delta} - g_n)^2}{\mu} + 2M \sum_{n=1}^{\infty} \left(\frac{(f_n^{\delta}(\tau) - f_n(\tau))^2}{\mu} \right) \right) \leq \\ &2(M+1) \frac{\delta^2}{\mu^2} \end{aligned}$$

故

$$\|\varphi^{\mu, \delta}(x) - \varphi^{\mu}(x)\| \leq \sqrt{2(M+1)} \frac{\delta}{\mu} \quad (7)$$

根据三角不等式,

$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq \|\varphi^{\mu, \delta}(x) - \varphi^{\mu}(x)\| + \|\varphi^{\mu}(x) - \varphi(x)\|$, 结合定理 1, 当 $\varphi(x)$ 满足条件 1), 则

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq C_1 E_1 \left(\frac{A(T)}{-\ln \mu} \right)^{\frac{p}{2}} (o(1) + 1) + \sqrt{2(M+1)} \frac{\delta}{\mu}$$

令 $\mu = \left(\frac{\delta}{E_1} \right) \left(\frac{1}{A(T)} \ln \frac{E_1}{\delta} \right)^{\frac{p}{2}}$, 当 $\delta \rightarrow 0^+$, 有

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq (\sqrt{2(M+1)} + C_1)$$

$$E_1 \left(\frac{1}{A(T)} \ln \frac{E_1}{\delta} \right)^{-\frac{p}{2}} (o(1) + 1)$$

当 $\varphi(x)$ 满足条件 2), 则

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq \begin{cases} \mu^{\frac{p}{2}} E_2 + \sqrt{2(M+1)} \frac{\delta}{\mu}, & 0 < p < 2 \\ C_2 \mu E_2 + \sqrt{2(M+1)} \frac{\delta}{\mu}, & p \geq 2 \end{cases}$$

故当 $0 < p < 2$ 时, 令 $\mu = \left(\frac{\delta}{E_2} \right)^{\frac{2}{p+2}}$, 有

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq (\sqrt{2(M+1)} + 1) \delta^{\frac{p}{p+2}} E_2^{\frac{2}{p+2}}$$

当 $p \geq 2$ 时, 令 $\mu = \left(\frac{\delta}{E_2} \right)^{\frac{1}{2}}$, 有

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq (\sqrt{2(M+1)} + C_2) E_2^{\frac{1}{2}} \delta^{\frac{1}{2}}$$

定理得证。

1.2 正则化参数的后验选取规则

应用偏差原理, 选择以下方程的解作为正则化参数:

$$\begin{aligned} \|\mu(K + \mu I)^{-1} (K \varphi^{\mu, \delta}(x) - \sum_{n=1}^{\infty} (g_n^{\delta} - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n^{\delta}(\tau) d\tau) \omega_n(x))\| &= \tau \delta \end{aligned}$$

其中 $\tau > \sqrt{2(M+1)}$ 是一个常数。

$$\begin{aligned} \text{引理 5} \quad \text{令 } \rho(\mu) &= \|\mu(K + \mu I)^{-1} (K \varphi^{\mu, \delta}(x) - \sum_{n=1}^{\infty} (g_n^{\delta} - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n^{\delta}(\tau) d\tau) \omega_n(x))\| \end{aligned}$$

$$\text{如果 } \left(\sum_{n=1}^{\infty} (g_n^{\delta} - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n^{\delta}(\tau) d\tau)^2 \right)^{\frac{1}{2}} > \tau \delta,$$

则有

(a) $\rho(\mu)$ 是一个连续函数;

(b) $\lim_{\mu \rightarrow 0} \rho(\mu) = 0$;

(c) $\lim_{\mu \rightarrow \infty} \rho(\mu) = \sqrt{\sum_{n=1}^{\infty} (g_n^{\delta} - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n^{\delta}(\tau) d\tau)^2}$;

(d) 对任意的 $\mu \in (0, \infty)$, $\rho(\mu)$ 是一个严格增函数。

证明: 与文[16]中引理 3.2.1 类似。

定理 3 令 $\varphi(x)$ 为问题(1)的精确解, $\varphi^{\mu, \delta}(x)$ 为正则化解, μ 为正则化参数; 如果存在常数 p 和 E_2 , 满足 $(\sum_{n=1}^{\infty} e^{\rho \lambda_n A(T)} \varphi_n^2)^{\frac{1}{2}} \leq E_2 < +\infty$, 有以下结论,

$$\|\varphi^{\mu, \delta}(x) - \varphi(x)\| \leq$$

$$\begin{cases} \left((\tau + \sqrt{2(M+1)})^{\frac{p}{p+2}} + \sqrt{2(M+1)} \cdot \left(\frac{1}{\tau - \sqrt{2(M+1)}} \right)^{\frac{2}{p+2}} \right) E_2^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}, & 0 < p < 2 \\ \left((\tau + \sqrt{2(M+1)})^{\frac{1}{2}} + \sqrt{2(M+1)} \cdot \left(\frac{C_2^2}{\tau - \sqrt{2(M+1)}} \right)^{\frac{1}{2}} \right) E_2^{\frac{1}{2}} \delta^{\frac{1}{2}}, & p \geq 2 \end{cases}$$

其中 C_2 为正常数。

证明 由三角不等式,

$$\|\varphi^{\mu,\delta}(x) - \varphi(x)\| \leq \|\varphi^{\mu,\delta}(x) - \varphi^\mu(x)\| + \|\varphi^\mu(x) - \varphi(x)\|$$

当 $0 < p < 2$ 时, 由 Hölder 不等式, 有

$$\begin{aligned} \|\varphi^\mu(x) - \varphi(x)\|^2 &= \left\| \sum_{n=1}^{\infty} \frac{-\mu}{\mu + e^{-\lambda_n A(T)}} \varphi_n \omega_n(x) \right\|^2 = \\ &= \sum_{n=1}^{\infty} \left(\left(\frac{\mu e^{-\lambda_n A(T)}}{\mu + e^{-\lambda_n A(T)}} \right)^{\frac{p}{2}} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^{1-\frac{p}{2}} \frac{\varphi_n}{(e^{-\lambda_n A(T)})^{\frac{p}{2}}} \right)^2 = \\ &= \sum_{n=1}^{\infty} \left(\frac{\mu e^{-\lambda_n A(T)}}{\mu + e^{-\lambda_n A(T)}} \right)^p \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^{\frac{2p-p^2}{p+2}} \left(\frac{\varphi_n}{(e^{-\lambda_n A(T)})^{\frac{p}{2}}} \right)^{\frac{2p}{p+2}} \times \\ &\quad \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^{\frac{4-2p}{p+2}} \left(\frac{\varphi_n}{(e^{-\lambda_n A(T)})^{\frac{p}{2}}} \right)^{\frac{4}{p+2}} \leq \\ &= \left(\sum_{n=1}^{\infty} \left(\frac{\mu e^{-\lambda_n A(T)}}{\mu + e^{-\lambda_n A(T)}} \right)^p \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^{\frac{2p-p^2}{p+2}} \times \right. \\ &\quad \left. \left(\frac{\varphi_n}{(e^{-\lambda_n A(T)})^{\frac{p}{2}}} \right)^{\frac{2p}{p+2}} \right)^{\frac{p}{p+2}} \times \\ &\quad \left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^{\frac{4-2p}{p+2}} \left(\frac{\varphi_n}{(e^{-\lambda_n A(T)})^{\frac{p}{2}}} \right)^{\frac{4}{p+2}} \right)^{\frac{p+2}{2}} \right)^{\frac{2}{p+2}} = \\ &= \left\| \sum_{n=1}^{\infty} \left(\frac{\mu e^{-\lambda_n A(T)}}{\mu + e^{-\lambda_n A(T)}} \right)^{\frac{p+2}{2}} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^{1-\frac{p}{2}} \frac{\varphi_n}{(e^{-\lambda_n A(T)})^{\frac{p}{2}}} \omega_n(x) \right\|^{\frac{2p}{p+2}} \times \\ &\quad \left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^{1-\frac{p}{2}} \frac{\varphi_n}{(e^{-\lambda_n A(T)})^{\frac{p}{2}}} \omega_n(x) \right\|^{\frac{4}{p+2}} \leq \\ &= \left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 \varphi_n e^{-\lambda_n A(T)} \omega_n(x) \right\|^{\frac{2p}{p+2}} \\ &= \left\| \sum_{n=1}^{\infty} \frac{\varphi_n}{(e^{-\lambda_n A(T)})^{\frac{p}{2}}} \omega_n(x) \right\|^{\frac{4}{p+2}} \leq \\ &= \left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 (g_n(x) - g_n^\delta(x) - \right. \\ &\quad \left. \int_0^T e^{-\lambda_n(A(T)-A(\tau))} (f_n(\tau) - f_n^\delta(\tau)) d\tau) \omega_n(x) \right\| + \\ &\quad \left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 \cdot (g_n^\delta(x) - \right. \\ &\quad \left. \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n^\delta(\tau) d\tau) \omega_n(x) \right\|^{\frac{2p}{p+2}} E_{\frac{4}{p+2}}^{\frac{4}{p+2}} \leq \\ &\quad (\tau + \sqrt{2(M+1)})^{\frac{2p}{p+2}} \delta^{\frac{2p}{p+2}} E_{\frac{4}{p+2}}^{\frac{4}{p+2}} \end{aligned}$$

即

(8)

当 $p \geq 2$ 时, 可以得到

$$\begin{aligned} \|\varphi^\mu(x) - \varphi(x)\|^2 &= \left\| \sum_{n=1}^{\infty} \frac{-\mu}{\mu + e^{-\lambda_n A(T)}} \varphi_n \omega_n(x) \right\|^2 = \\ &= \sum_{n=1}^{\infty} \left(\frac{\mu e^{-\lambda_n A(T)}}{\mu + e^{-\lambda_n A(T)}} \frac{\varphi_n}{e^{-\lambda_n A(T)}} \right)^2 = \sum_{n=1}^{\infty} \left(\frac{\mu e^{-\lambda_n A(T)}}{\mu + e^{-\lambda_n A(T)}} \right)^2 \\ &= \frac{\varphi_n}{e^{-\lambda_n A(T)}} \frac{\varphi_n}{e^{-\lambda_n A(T)}} \leq \\ &= \left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu e^{-\lambda_n A(T)}}{\mu + e^{-\lambda_n A(T)}} \right)^2 \frac{\varphi_n}{e^{-\lambda_n A(T)}} \right)^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left(\frac{\varphi_n}{e^{-\lambda_n A(T)}} \right)^2 \right)^{\frac{1}{2}} \leq \\ &= \left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 \varphi_n e^{-\lambda_n A(T)} \right)^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \left(\frac{\varphi_n}{e^{-\lambda_n A(T)}} \right)^2 \right)^{\frac{1}{2}} = \\ &= \left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 \cdot (g_n - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau) \omega_n(x) \right\| \cdot \\ &\quad \left(\sum_{n=1}^{\infty} e^{2\lambda_n A(T)} \varphi_n^2 \right)^{\frac{1}{2}} \leq \left(\left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 (g_n - g_n^\delta - \right. \right. \\ &\quad \left. \left. \int_0^T e^{-\lambda_n(A(T)-A(\tau))} (f_n(\tau) - f_n^\delta(\tau)) d\tau) \omega_n(x) \right\| + \right. \\ &\quad \left. \left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 (g_n^\delta - \right. \right. \\ &\quad \left. \left. \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n^\delta(\tau) d\tau) \omega_n(x) \right\| \right) E_2 \leq \\ &\quad (\delta \sqrt{2(M+1)} + \tau \delta) E_2 \end{aligned}$$

故有

$$\|\varphi^\mu(x) - \varphi(x)\| \leq (\tau + \sqrt{2(M+1)})^{\frac{1}{2}} E_2^{\frac{1}{2}} \delta^{\frac{1}{2}} \quad (9)$$

下面对 $\frac{1}{\mu}$ 进行估计:

$$\begin{aligned} \tau \delta &\leq \left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 ((g_n - g_n^\delta) - \right. \\ &\quad \left. \int_0^T e^{-\lambda_n(A(T)-A(\tau))} (f_n(\tau) - f_n^\delta(\tau)) d\tau) \omega_n(x) \right\| + \\ &= \left\| \sum_{n=1}^{\infty} \left(\frac{\mu}{\mu + e^{-\lambda_n A(T)}} \right)^2 (g_n - \int_0^T e^{-\lambda_n(A(T)-A(\tau))} f_n(\tau) d\tau) \omega_n(x) \right\| \leq \\ &= \left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu (e^{\lambda_n A(T)})^{\frac{2-p}{4}}}{1 + \mu e^{\lambda_n A(T)}} \right)^2 (e^{-\lambda_n A(T)})^{-\frac{p}{2}} \varphi_n \right)^2 \right)^{\frac{1}{2}} + \\ &\quad \sqrt{2(M+1)} \delta \end{aligned}$$

当 $0 < p < 2$ 时, 由引理 1 得

$$\left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu (e^{\lambda_n A(T)})^{\frac{2-p}{4}}}{1 + \mu e^{\lambda_n A(T)}} \right)^2 (e^{-\lambda_n A(T)})^{-\frac{p}{2}} \varphi_n \right)^2 \right)^{\frac{1}{2}} \leq \mu^{\frac{2+p}{2}} E_2$$

当 $p \geq 2$ 时, 根据引理 2, 得

$$\left(\sum_{n=1}^{\infty} \left(\left(\frac{\mu (e^{\lambda_n A(T)})^{\frac{2-p}{4}}}{1 + \mu e^{\lambda_n A(T)}} \right)^2 (e^{-\lambda_n A(T)})^{-\frac{p}{2}} \varphi_n \right)^2 \right)^{\frac{1}{2}} \leq E_2 (C_2 \mu)^2$$

因此,

$$\frac{1}{\mu} \leq \begin{cases} \left(\frac{1}{\tau - \sqrt{2(M+1)}} \right)^{\frac{2}{2+p}} \left(\frac{E_2}{\delta} \right)^{\frac{2}{2+p}}, & 0 < p < 2 \\ \left(\frac{C_2^2}{\tau - \sqrt{2(M+1)}} \right)^{\frac{1}{2}} \left(\frac{E_2}{\delta} \right)^{\frac{1}{2}}, & p \geq 2 \end{cases} \quad (10)$$

联合式(7)、(8)、(9) 和(10), 可得到定理结论。

2 数值实验

本节将列出两个数值算例来显示方法的可行性。测量数据是通过添加随机扰动生成

$$g^\delta = g + \varepsilon g(2 \cdot \text{rand}(\text{size}(g)) - 1)$$

$$f^\delta = f + \varepsilon f(2 \cdot \text{rand}(\text{size}(f)) - 1)$$

则误差水平 $\delta = \max\{\varepsilon \|g\|, \varepsilon \|f\|\}$ 。

例 1 令 $T = 1, L = \pi, a(t) = 2t + 1, g(x) = u(x, 1) = \frac{x \sin x}{e^2}, f(x, t) = \frac{-2(2t + 1) \cos x}{e^{t^2+t}}$, 则反初值问题的解为 $u(x, 0) = \varphi(x) = x \sin x$ 。

例 2 令 $T = 1, L = \pi, a(t) = 2t + 1, g(x) = u(x, 1) = \frac{2 \sin 2x}{e^2}, f(x, t) = \frac{6(2t + 1) \sin 2x}{e^{t^2+t}}$, 反初值问题的解为 $u(x, 0) = \varphi(x) = 2 \sin 2x$ 。

例 1 的解满足 $(\sum_{n=1}^{\infty} \lambda_n^p \varphi_n^2)^{\frac{1}{2}} \leq E_1 < +\infty$, 如图 1 所示, 本文只画出正则化参数先验选择规则下正则化解和精确解的图象; 例 2 的解满足 $(\sum_{n=1}^{\infty} e^{p \lambda_n A(T)} \varphi_n^2)^{\frac{1}{2}} \leq E_2 < +\infty$, 图 2 中画出了先验和后验正则化参数选择规则下近似解和精确解的图象; 两个例子表明本文提出的正则化参数选取策略是有效的。

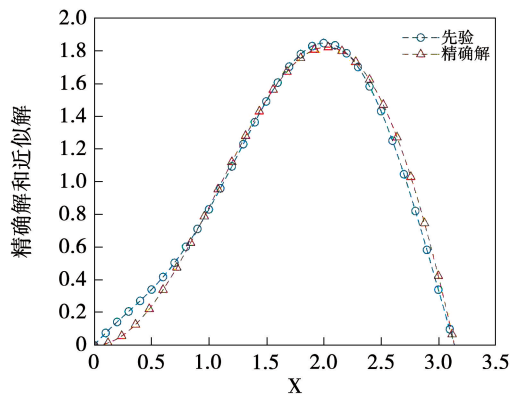


图 1 例 1 $\varepsilon = 0.001, p = 2$ 正则化解和精确解的比较
Fig.1 The comparison of numerical effects between the exact solution and its regularization solution for Example 1: $\varepsilon = 0.001, p = 2$

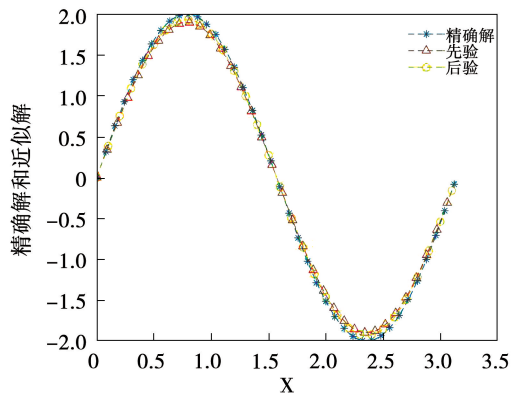


图 2 例 2 $\varepsilon = 0.001, p = 2$ 正则化解和精确解的比较
Fig.2 The comparison of numerical effects between the exact solution and its regularization solution for Example 2: $\varepsilon = 0.001, p = 2$

3 结 论

本文考虑了带有时间依赖系数的一维热传导方程的反初值问题, 采用拟边值方法构造正则化解, 并分别给出了正则化参数的先验和后验选取准则, 最后用数值算例验证了拟边值方法的有效性。

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(编辑:温泽宇)